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ASYMPTOTIC SOLUTIONS FOR THE UNSTEADY GRAETZ PROBLEM

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Abstract—Heat transfer to a Poiseuille flow in a tube is studied under transient conditions, resulting from a step increase in wall temperature at time $\theta = 0$ for $z > 0$, where z is the axial distance. Asymptotic solutions for small time or large axial distance, and for small axial distance, are obtained. In the former case, the solution is seen to be exactly independent of axial distance. In the latter case, the solution is independent of time for $z \ll \theta$, and independent of axial position for $z \gg \theta$; between these two limits, the solution is a function of both time and axial position.

NOMENCLATURE

$Ai(x)$,	Airy function of the first kind;
C,	heat capacity;
erfc(x),	complementary error function;
E,	constant in equation (8);
${}_1F_1$,	confluent hypergeometric function;
H,	unit step function;
k,	thermal conductivity;
R,	tube radius;
s, p,	Laplace transform variables related to ϕ and x, respectively;
t,	temperature;
t_0,	temperature at entrance of heated section;
t_w,	wall temperature;
T,	dimensionless temperature $= (t - t_0)/(t_w - t_0)$;
T^*,	double Laplace transform of T;
V,	velocity in axial direction;
V_{max},	maximum fluid velocity;
w,	$1 + y$;
x,	dimensionless axial distance $= \dfrac{kz}{\rho C V_{max} R^2}$;
y,	dimensionless radial distance $= r/R$;
z, r,	axial and radial distances, respectively.

Greek symbols

$\Gamma(x)$,	gamma function;
ρ,	density;
θ,	time;
ϕ,	dimensionless time $= k\theta/\rho C R^2$;
σ,	defined by equation (9);
Δ,	defined by equation (10).

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INTRODUCTION

TRANSIENT heat transfer to a fully developed laminar flow between flat plates and in a circular tube has been treated by Sparrow and Siegel [1, 2]. The energy equation was integrated over the cross section, the temperature distribution written as a third order polynomial, and the resulting equation solved by the method of characteristics. The results show an abrupt change from a transient solution which is independent of position in the direction of flow to a steady-state solution, with a change in temperature slope at this point. The same authors attack the unsteady turbulent case in much the same manner [3], again resulting in a sudden change from transient to steady-state solutions.

Improvements on the above solutions were made by Siegel [4] for laminar flow, and by Sparrow and Siegel [3] for turbulent flow. A series expansion is made around the steady-state solution. The energy equation is integrated over the cross sectional area, and the method of characteristics used to determine the terms of the series. The heat flux is now found to vary smoothly between the transient and the steady-state. The solution is exact for large time, and is a good approximation for small time.

On the other hand, recent advances in the asymptotic theory of linear differential equations of second order for large values of a parameter have made practical direct integral transform approaches to this and similar problems [5–11, 12]. In this note the problem is attacked by means of a double Laplace transform. Asymptotic solutions are found for large values of each of the transform variables, and these asymptotic solutions are then inverted. The results are, to the accuracy of a graphical presentation, identical to a numerical solution given by Sparrow and Siegel [1].

EQUATIONS

The dimensionless equation describing unsteady heat transfer to fully developed incompressible laminar Newtonian flow in a tube is

$$\frac{\partial T}{\partial \phi} + (1 - y^2)\frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial y^2} + \frac{1}{y}\frac{\partial T}{\partial y}. \qquad (1)$$

It is assumed that the Poiseuille velocity profile is undisturbed by temperature changes, that the properties of the fluid, i.e. thermal conductivity, density, and heat capacity, are constant, and that axial conduction may be neglected. If the fluid is initially everywhere at the same temperature, and at some time a step temperature is imposed on the wall of the tube, the dimensionless initial and boundary conditions are:

$$\begin{array}{lll} \phi = 0 & T = 0 & \\ x = 0 & T = 0 & \\ y = 0 & T = \text{finite} & \\ y = 1 & T = 1 & x > 0, \phi > 0. \end{array}$$

The problem will be attacked by use of the double Laplace transform

$$T^* = \int_0^\infty \int_0^\infty T \exp[-s\phi - px]\, d\phi\, dx$$

reducing the equation to

$$\frac{d^2T^*}{dy^2} + \frac{1}{y}\frac{dT^*}{dy} + (-s - p + y^2 p)T^* = 0 \quad (2)$$

with boundary conditions

$$\begin{array}{ll} y = 1 & T^* = 1/sp \\ y = 0 & T^* \ \text{finite}. \end{array}$$

The solution to the above system may be written as

$$T = \frac{1}{(2\pi i)^2}\int_{C-i\infty}^{C+i\infty}\int_{C'-i\infty}^{C'+i\infty} \frac{1}{sp} \frac{\exp\left[-\frac{ip^{\frac{1}{2}}y^2}{2}\right] {}_1F_1\left(\frac{1}{2} + \frac{s+p}{4ip^{\frac{1}{2}}}; 1; ip^{\frac{1}{2}}y^2\right) \exp[s\phi + px]}{\exp\left[-\frac{ip^{\frac{1}{2}}}{2}\right] {}_1F_1\left(\frac{1}{2} + \frac{s+p}{4ip^{\frac{1}{2}}}; 1; ip^{\frac{1}{2}}\right)}\, ds\, dp \qquad (3)$$

where ${}_1F_1$ is the confluent hypergeometric function. Details of this and of later steps can be found in (13).

ASYMPTOTIC SOLUTION FOR SMALL TIME

Asymptotic expressions for large p and for large s, corresponding to small axial distance and to small time respectively, will now be used to evaluate equation (3). It may be shown (13), that for large s equation (3) can be written

$$T = \frac{1}{(2\pi i)^2}\int_{C-i\infty}^{C+i\infty}\int_{C'-i\infty}^{C'+i\infty} \frac{1}{sp} \frac{\exp[-(s+p)^{\frac{1}{2}}(1-y) + s\phi + px]}{y^{\frac{1}{2}}} \left\{ \frac{1 + \frac{3 - 4py^4}{24y(s+p)^{\frac{1}{2}}} + 27 + 88py^4 + \frac{16}{3}p^2y^8 + \dots}{1 + \frac{3 - 4p}{24(s+p)^{\frac{1}{2}}} + \frac{27 + 88p + \frac{16}{3}p^2}{4 . 16 . 6 (s+p)} + \dots} \right\} ds\, dp. \qquad (4)$$

The temperature can now be found from equation (4) by dividing the two asymptotic series and inverting term by term. The result is

$$\left.\begin{aligned} T = &\left\{\frac{1}{y^{\frac{1}{2}}}\left[H(\phi - x)\operatorname{erfc}\frac{1-y}{2x^{\frac{1}{2}}} + H(x-\phi)\operatorname{erfc}\frac{1-y}{2\phi^{\frac{1}{2}}}\right]\right\} + \\ &\left\{\frac{1-y}{4y^{3/2}}\left[H(x-\phi)\,\phi^{\frac{1}{2}}\, i\operatorname{erfc}\frac{1-y}{2\phi^{\frac{1}{2}}} + H(\phi - x)\, x^{\frac{1}{2}}\, i\operatorname{erfc}\frac{1-y}{2x^{\frac{1}{2}}}\right] + \right.\\ &\left.\frac{y-y^4}{6y^{3/2}}\,\frac{H(\phi-x)}{\pi^{\frac{1}{2}}x^{\frac{1}{2}}}\exp\left[-\frac{(1-y)^2}{4x}\right]\right\} + \left\{\frac{(9-2y-7y^2)}{32y^{3/2}}\left[H(x-\phi)\phi\, i^2\operatorname{erfc}\frac{1-y}{2\phi^{\frac{1}{2}}} + \right.\right.\\ &\left. H(\phi-x)x\, i^2\operatorname{erfc}\frac{1-y}{2x^{\frac{1}{2}}}\right] + \frac{1-13y^2+11y^4+y^5}{48y^{5/2}}H(\phi-x)\operatorname{erfc}\frac{1-y}{2x^{\frac{1}{2}}} + \\ &\left.\frac{y^2-2y^5+y^8}{72y^{5/2}}\,\frac{H(\phi-x)}{2}\,\frac{1-y}{\pi^{\frac{1}{2}}x^{3/2}}\exp\left[-\frac{(1-y)^2}{4x}\right]\right\} + \ldots; \\ &H(\eta) = \begin{cases}1 & \eta > 0; \\ 0 & \eta < 0;\end{cases} \quad i^n\operatorname{erfc}\eta = \int_\eta^\infty i^{n-1}\operatorname{erfc} t\, dt \quad n = 0, 1, 2, \ldots. \end{aligned}\right\} \quad (5)$$

Equation (5) holds for $y > 0$. A similar expression, valid at the center of the tube, is presented in (13). It is also shown there that equation (5) is a series of terms of ascending order in ϕ. It can be shown that equation (5) satisfies the initial and boundary conditions of the problem (13).

ASYMPTOTIC SOLUTION FOR SMALL AXIAL DISTANCE

For large p, or small axial distance, the curvature of the tube may be neglected. Also, since in the region of developing temperature profile the temperature changes are confined to a region near the wall, the velocity profile $v = 2w - w^2$ may be replaced by

$$v = 2w + \sum_{n=2}^{N} a_n w^n \qquad 0 \leqslant w < \infty \qquad (6)$$

where w is now the distance measured from the wall, i.e. $w = 1 + y$. Equation (6) can be made to approximate closely the true velocity profile near the wall, but must have its only zero at $w = 0$. There is no longer a zero at $w = 2$. With these assumptions equation (2) becomes

$$\frac{d^2T^*}{dw^2} - sT^* - p(2w + \sum_{n=2}^{N} a_n w^n)\, T^* = 0 \quad (7)$$

with

$$w = 0 \qquad T^* = 1/sp$$
$$w = \infty \qquad T^* \quad \text{finite}$$

An aymptotic solution of equation (7) can be found for large p by the method of Langer [8]. Omitting the details, the result is

$$\left.\begin{aligned} T^*(s, p, w) = E&\left[Ai(\zeta)\left\{(\sigma')^{\frac{1}{2}} + \frac{(\sigma')^{\frac{1}{2}}}{2p}\int_0^W [\sigma'(t)]^{\frac{1}{2}}\,\Delta(t)\,dt + \ldots\right\}\frac{(2p)^{1/3}(\sigma')^{\frac{1}{2}}}{4p\sigma^{\frac{1}{2}}}\right.\\ &\left. Ai'(\zeta)\left\{\int_0^w \frac{s\sigma'(t)[1-(\sigma'(t))^2]\,dt}{[\sigma(t)]^{\frac{1}{2}}} + \ldots\right\}\right] \end{aligned}\right\} \quad (8)$$

where E is an arbitrary constant to be found from the boundary condition at $w = 0$,

$$\sigma(w) = \left[\frac{3}{2}\int_0^w (2w + \sum_{n=2}^{N} a_n w^n)^{\frac{1}{2}}\,dw\right]^{2/3} \qquad (9)$$

$$\Delta(w) = \frac{s(\sigma')^{\frac{1}{2}}}{4\sigma^{\frac{1}{2}}}[1 - s + (\sigma')^2 s]\int_0^w \sigma^{\frac{1}{2}}(t)\,[(\sigma'(t))^2 - 1]\,dt \qquad (10)$$

Ai = Airy function

$$\zeta = \frac{s + 2p\sigma}{(2p)^{2/3}} \tag{11}$$

The primes denote differentiation with respect to the argument. The heat flux at the wall can be obtained by differentiating equation (8) and determining the double inverse transform. The first term in the series for the gradient at the wall is

$$\left.\frac{\mathrm{d}T}{\mathrm{d}w}\right|_{w=0} = -\frac{3^{5/6}\,\Gamma(\frac{2}{3})}{2^{2/3}\pi\, x^{1/3}} - \sum_{n} \frac{3}{2^{4/3}\,\pi\, b_n\, x^{1/3}} \int_0^\infty \exp\left(-\, 2^{-\frac{1}{2}}\, \xi^{3/2}\right) \sin\left(\xi^{3/2}\, 2^{-\frac{1}{2}} - \frac{b_n\, \phi\, \xi 2^{\frac{2}{3}}}{x^{\frac{2}{3}}} + \frac{\pi}{4}\right) \frac{\mathrm{d}\xi}{\xi^{\frac{1}{2}}} \tag{12}$$

where b_n are the zeros of Ai.

DISCUSSION

These results will be compared with work of Siegel and Sparrow for transient heat transfer to laminar forced convective flow between flat plates and in a tube. Our expression for the heat flux for small axial distance holds for either plates or tubes since wall curvature may be neglected in this region. For small time, or large axial distance, we have shown for the tube that the solution to the problem reduces to that of conduction. It may be shown, but is not done here, that a similar result arises in the case of flow between flat plates. It is thus possible to construct the entire solution for both cases.

In Fig. 1, the flat plate dimensionless temperature gradient is shown as a function of dimensionless time, with dimensionless distance as a parameter. Three solutions are shown: the solutions of Sparrow and Siegel found by the method of characteristics and by finite differences, and the results of the present work. The latter are, to the accuracy of the graph, identical to the exact finite difference solution.

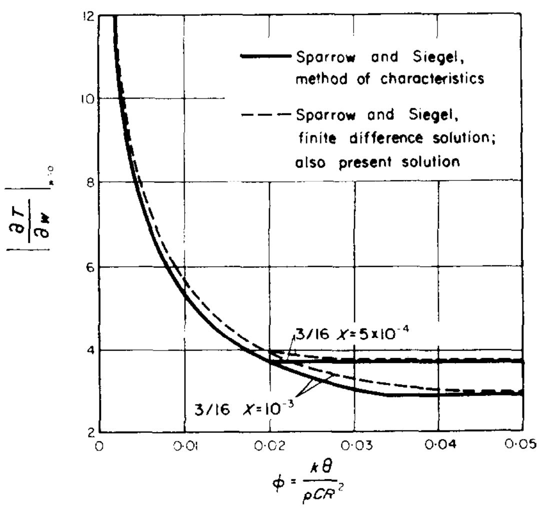

FIG. 1. Transient dimensionless temperature gradient at the wall.

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Résumé—Le transport de chaleur dans un écoulement de Poiseuille à l'intérieur d'un tube est étudié sous des conditions transitoires, résultant d'un accroissement brutal de la température pariétale à l'instant $\theta = 0$ pour $z > 0$, où z est la distance axiale. On obtient des solutions asymptotiques pour un temps faible ou une grande distance axiale, et pour une petite distance axiale. Dans le premier cas, on voit que la solution est rigoureusement indépendante de la distance axiale. Dans le deuxième cas, la solution est indépendante du temps pour $z \ll \theta$, et indépendante de la position sur l'axe pour $z \gg \theta$; entre ces deux limites, la solution est fonction à la fois du temps et de la position sur l'axe.

Zusammenfassung—Die Wärmeübertragung an eine laminare Strömung in einem Rohr wird bei verschiedenen Bedingungen untersucht. Der Wärmeübergang rührt von einer stufenweisen Steigerung der Wandtemperatur bei der Zeit $\theta = 0$ für $z > 0$ her, wobei z die axiale Entfernung bedeutet. Asymptotische Lösungen für einen kleinen Zeitabschnitt oder einer grossen axialen Entfernung und für eine kleine axiale Entfernung werden erzielt. In dem ersteren Fall ist die Lösung vollkommen unabhängig von der axialen Entfernung. Im letzteren Fall ist das Ergebnis unabhängig von der Zeit für $z \ll \theta$ und unabhängig von der axialen Lage für $z \gg \theta$. Zwischen diesen beiden Grenzen ist die Lösung eine Funktion sowohl der Zeit als auch der axialen Entfernung.

Аннотация—Исследуется задача о теплообмене при пуазейлевском течении в трубе в случае переходного процесса, возникающего в результате скачкообразного повышения температуры стенки в момент времени $\theta = 0$ при $z > 0$, где z—расстояние вдоль оси. Получены асимптотические решения для малых времен или больших осевых расстояниях, а также для небольших расстояний по оси. Установлено, что в первом случае решение совершенно не зависит от расстояния вдоль оси. Во втором случае решение не зависит от времени при $z \ll \theta$ и не зависит от положения на оси для $z \gg \theta$. В промежутке между этими предельными случаями решение есть функция времени и расстояния вдоль оси.